EXERCISES 8.1

1. *a. associative *b. commutative c. neither
   d. commutative, associative e. commutative

*2. a. not commutative: $a \cdot b \neq b \cdot a$
   not associative: $a \cdot (b \cdot d) \neq (a \cdot b) \cdot d$

   b.

·	p	q	r	s
p	p	q	r	s
q	q	r	s	p
r	r	s	p	q
s	s	p	q	r

commutative

3. For example:
   a. $a \cdot b = (a + b)^2$ b. $a \cdot b = a$
   c. $a \cdot b = a + 1$ d. $a \cdot b = a + b$

4. Compute $\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \cdot \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right)$$

The results are equal by the distributive, commutative, and associative properties of **Z** under addition and multiplication.

5. *a. Semigroup
   *b. Not a semigroup - not associative
   *c. Not a semigroup - S not closed under ·
   *d. Monoid; $i = 1 + 0\sqrt{2}$
   *e. Group; $i = 1 + 0\sqrt{2}$
   *f. Group; i = 1
   *g. Monoid; i = 1
   h. Monoid; i = 1
   i. Semigroup
   j. Monoid; i = (0,1)
   k. Group; i = 0
   l. Not a semigroup - S not closed under ·
   m. Group; i = zero polynomial
   n. Not a semigroup - S not closed under ·
   o. Group; $i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   p. Group; i = 1
   q. Group; i = 0
   r. Monoid; i = function mapping every x to 0

*6. a. f_0 = identity function $f_1(1) = 2$ $f_1(2) = 1$
$f_2(1) = 1$ $f_2(2) = 1$ $f_3(1) = 2$ $f_3(2) = 2$

$\circ$	f_0	f_1	f_2	f_3
f_0	f_0	f_1	f_2	f_3
f_1	f_1	f_0	f_3	f_2
f_2	f_2	f_2	f_2	f_2
f_3	f_3	f_3	f_3	f_3

b. f_0 and f_1 are the elements

$\circ$	f_0	f_1
f_0	f_0	f_1
f_1	f_1	f_0

*7.

$\circ$	R_1	R_2	R_3	F_1	F_2	F_3
R_1	R_2	R_3	R_1	F_3	F_1	F_2
R_2	R_3	R_1	R_2	F_2	F_3	F_1
R_3	R_1	R_2	R_3	F_1	F_2	F_3
F_1	F_2	F_3	F_1	R_3	R_1	R_2
F_2	F_3	F_1	F_2	R_2	R_3	R_1
F_3	F_1	F_2	F_3	R_1	R_2	R_3

Identity element is R_3; inverse for F_1 is F_1; inverse for R_2 is R_1.

8. $\alpha_1 \to R_3$, $\alpha_2 \to F_3$, $\alpha_3 \to F_2$, $\alpha_4 \to F_1$, $\alpha_5 \to R_1$, $\alpha_6 \to R_2$

9. *a. No - not the same operation
*b. No - zero polynomial (identity) does not belong to P
c. No - not every element of Z has an inverse in Z*
d. Yes
e. No - Z is not a subset of $M_2(Z)$
f. Yes
g. No - {0,3,6} not closed under $+_8$

*10. $[\{0\},+_{12}]$, $[Z_{12},+_{12}]$, $[\{0,2,4,6,8,10\},+_{12}]$, $[\{0,4,8\},+_{12}]$, $[\{0,3,6,9\},+_{12}]$, $[\{0,6\},+_{12}]$

11. In each set closure holds, i is a member, and each element has an inverse.

*12. 4!/2 = 24/2 = 12 elements

$\alpha_1 = i$ $\alpha_2 = (1,2)o(3,4)$ $\alpha_3 = (1,3)o(2,4)$ $\alpha_4 = (1,4)o(2,3)$
$\alpha_5 = (1,3)o(1,2)$ $\alpha_6 = (1,2)o(1,3)$ $\alpha_7 = (1,3)o(1,4)$ $\alpha_8 = (1,4)o(1,2)$
$\alpha_9 = (1,4)o(1,3)$ $\alpha_{10} = (1,2)o(1,4)$ $\alpha_{11} = (2,4)o(2,3)$ $\alpha_{12} = (2,3)o(2,4)$

is a bijection. It is also a homomorphism. For x, y elements of $\mathbb{Z}_5$,
$$f(x + y) = [x + y] = [x] + [y] = f(x) + f(y)$$

c. The inverse of [10] is [-10] = [4].
The pre-image of [10] is 10.

EXERCISES 8.2

*1. a. 0001111110 b. aaacaaaa c. 00100110

2. a. 110100, 111010 b. none c. $0a_1a_2a_3a_4a_5$

3.

	Next state		
	Present input		
Present state	0	1	Output
s_0	s_1	s_1	0
s_1	s_2	s_1	1
s_2	s_2	s_0	0

Output is 010010

4.

	Next state		
	Present input		
Present state	0	1	Output
s_0	s_3	s_0	1
s_1	s_2	s_0	0
s_2	s_2	s_1	1
s_3	s_1	s_2	0

Output is 11101011

*5.

	Next state		
	Present input		
Present state	0	1	Output
s_0	s_1	s_2	a
s_1	s_2	s_3	b
s_2	s_2	s_1	c
s_3	s_2	s_3	b

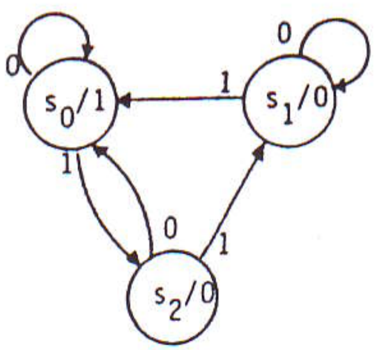
Output is abbcbb

6.

	Next state			
	Present input			
Present state	a	b	c	Output
s_0	s_0	s_1	s_2	0
s_1	s_0	s_2	s_2	1
s_2	s_0	s_2	s_1	0

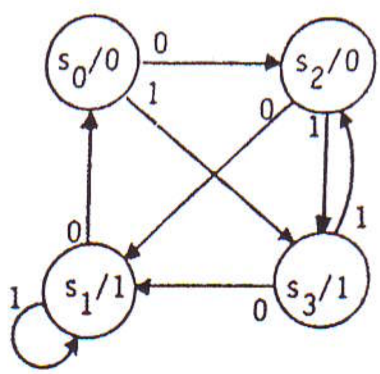
Output is 0000010

7.



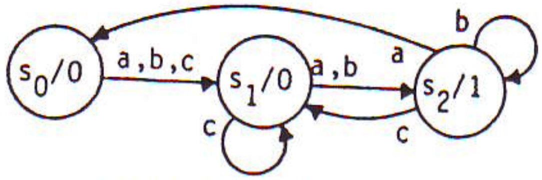
Output is 101110

8.



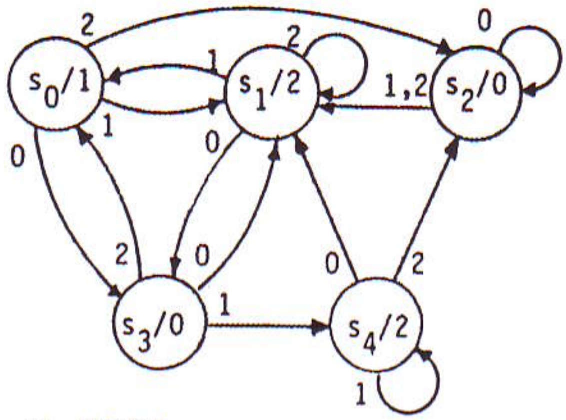
Output is 00111

*9.



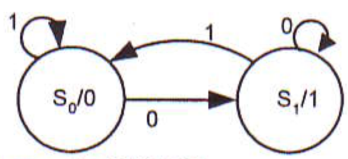
Output is 0001101

10.



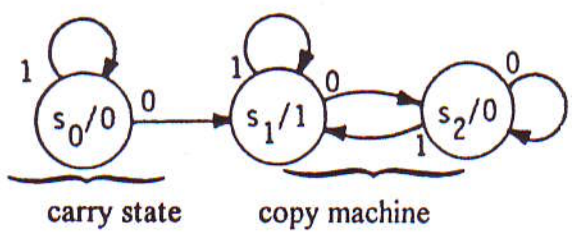
Output is 102012

*11. a.

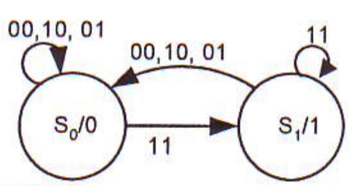


b. Output is 010100

12.

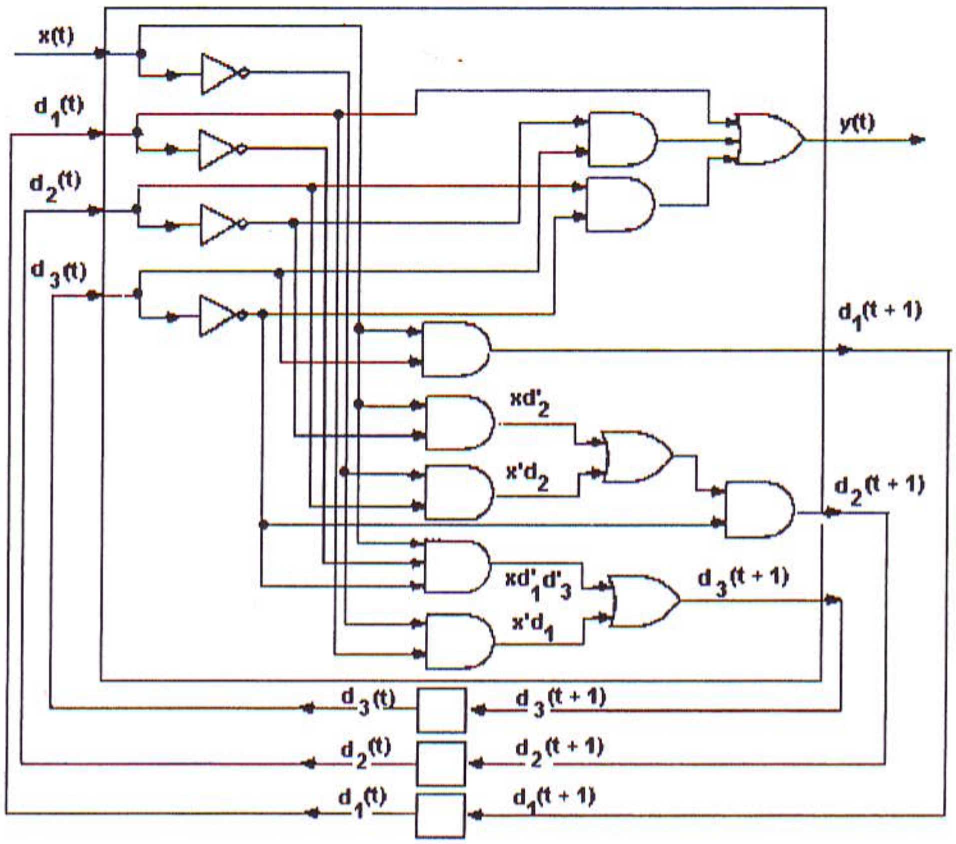


13. a.



b. 010010

$y(t) = d_1 + d_2d_3' + d_2'd_3$
$d_1(t + 1) = xd_3$
$d_2(t + 1) = d_3'(xd_2' + x'd_2)$
$d_3(t + 1) = xd_1'd_3' + x'd_1$



EXERCISES 8.3

*1. a. halts with final tape

···	*b*	0	0	0	0	0	*b*	···

b. does not change the tape and moves forever to the left

2. a. halts with final tape

···	*b*	1	0	1	0	*b*	···

b. moves forever back and forth adding a 1 at each end of the nonblank portion of the tape.

3. (0,0,1,0,R)
(0,1,0,0,R)

4. One answer: State 1 is a final state

(0,0,0,0,R)	passes over 0's
(0,1,1,1,R)	first 1, go to final state, halt and accept

*5. *One answer:* *State 2 is a final state*

(0,b,b,2,R)	blank tape or no more 1's
(0,1,1,1,R)	has read odd number of 1's
(1,1,1,0,R)	has read even number of 1's

6. One answer: State 3 is a final state

(0,0,0,0,R) (0,1,1,1,R)	pass over 0's to first 1
(1,0,0,1,R) (1,1,1,2,R)	pass over 0's to second 1
(2,b,b,3,R)	end of string, halt and accept

7. One answer: State 3 is a final state

(0,(,(,0,R) (0,),X,1,L) (0,X,X,0,R)	looks for leftmost), replaces it with X
(0,b,b,2,L)	no more)
(1,(,X,0,R) (1,X,X,1,L)	looks for matching (
(2,b,b,3,R) (2,X,X,2,R)	no more (, halts and accepts

*8. One answer: State 9 is a final state

(0,b,b,9,R)	accepts blank tape
(0,0,0,0,R) (0,1,X,1,R)	finds first 1, marks with X
(1,1,1,1,R) (1,Y,Y,1,R)	searches right for 2's
(1,2,Y,3,R) (3,2,Y,4,L)	pair of 2's, marks with Y's

(4,Y,Y,4,L) (4,X,X,4,L) (4,1,1,4,L) (4,Z,Z,4,L)	searches left for 0's
(4,0,Z,5,L) (5,0,Z,6,R)	pair of 0's, marks with Z's
(6,Z,Z,6 R) (6,X,X,6,R) (6,1,X,1,R)	passes right to next 1
(6,Y,Y,7,R)	no more l's
(7,Y,Y,7,R) (7,b,b,8,L)	no more 2's
(8,Y,Y,8,L) (8,X,X,8,L) (8,Z,Z,8,L) (8,b,b,9,L)	no more 0's, halts and accepts

9. One answer: State 7 is a final state

(0,0,b,1,R)	0 is leftmost symbol
(1,0,0,1,R) (1,1,1,1,R) (1,*,*,1,R) (1,b,b,2,L)	finds right end
(2,0,b,3,L)	match, erases right symbol
(3,0,0,3,L) (3,1,1,3,L) (3,*,*,3,L) (3,b,b,0,R)	finds left end and begins again
(0,1,b,4,R)	1 is leftmost symbol
(4,0,0,4,R) (4,1,1,4,R) (4,*,*,4,R) (4,b,b,5,R)	finds right end
(5,1,b,3,L)	match, erases right symbol
(0,*,*,6,R)	word left of * is empty
(6,b,b,7,R)	word right of * is empty, halts and accepts

EXERCISES 8.4

*1. a. L(G) = {a}
  b. L(G) = {010101, 010111, 011101, 011111, 110101, 110111, 111101, 111111}
  c. L(G) = 0(10)*
  d. L(G) = 0*1111*

2. a. (c) is regular, (b) and (d) are context-free
  b. For example:
  for (a): G = (V, V_T, S, P) where V = {a, S}, V_T = {a}, and P = {S → a}.
  for (b): G = (V, V_T, S, P) where V = {0, 1, A, B, C, D, E, , G, S}, V_T = {0, 1}, and P consists of the productions

S → 0A	C → 1D
S → 1A	D → 1E
A → 1B	D → 1F
A → 1C	E → 0G
B → 0D	F → 1G
	G → 1

  for (d): G = (V, V_T, S, P) where V = {0, 1, A, B, S}, V_T = {0, 1}, and P consists of the productions

S → 0S	B → 1B
S → 1A	B → 1
A → 1B	

3. L(G) = (ab)*

4. L(G) = {1^n01^n | n ≥ 0}

*5. L(G) = aa*bb*. G is context-sensitive. An example of a regular grammar that generates L(G) is G' = (V, V_T, S, P) where V = {a, b, A, B, S}, V_T = {a, b}, and P consists of the productions

S → aA	A → aA	B → bB
S → aB	A → aB	B → b

6. a. <S> : : = λ | 0<A>
  <A>: : = 0<B>
  <B>: : = 0 | 0<C>
  <C>: : = 0<B>
  b. <S> : : = 0<A> | 1<A>
  <A>: : = 1<B><B>
  <B>: : = 01| 11

c. <S> : : = 0 | 0<A>
<A> : : = 1<B>
<B> : : = 0<A> | 0

d. <S> : : = 0<S> | 11<A>
<A> : : = 1<A> | 1

7. a. 0*011*
b. $S \to 0S$
$S \to 0A$
$A \to 1$
$A \to 1A$

*8. a. 1*1(00)*
b. $S \to 1$
$S \to 1S$
$S \to 0A$
$A \to 0$
$A \to 0B$
$B \to 0A$

9. a. appleyay
b. onkeymay
c. ainchay
d. For example, $G = (V, V_T, S, P)$ where
$V = \{a, e, i, o, u, b, c, d, f, \ldots x, y, z, O, L, C, S\}$
$V_T = \{ a, e, i, o, u, b, c, d, f, \ldots x, y, z\}$
and P consists of the productions

$S \to OLyay$
$S \to OLCay$
$L \to \lambda$ $L \to LL$ $L \to C$ $L \to O$
$O \to a \mid e \mid i \mid o \mid u$
$C \to CC \mid b \mid c \mid d \mid f \mid \ldots \mid x \mid y \mid z$

e. $S \to OLCay \to aLCay \to aLLCay \to aOLCay \to aiLCay \to aiCCay$
$\to ainCay \to ainCCay \to aincCay \to ainchay$

10. For example, $G = (V, V_T, S, P)$ where $V = \{(,), S\}$, $V_T = \{(,)\}$, and P consists of the productions
$S \to \lambda$
$S \to (S)S$