- Find least, minimal, greatest, and maximal elements in a partially ordered set.
- W Find the equivalence classes associated with an equivalence relation.

MAIN IDEAS

- A binary relation on a set S is formally a subset of $S \times S$; the distinctive relationship satisfied by the relation's members often has a verbal description as well.
- Operations on binary relations on a set include union, intersection, and complementation.
- Binary relations can have properties of reflexivity, symmetry, transitivity, and antisymmetry.
- Finite partially ordered sets can be represented graphically.
- An equivalence relation on a set defines equivalence classes, which may themselves be treated as entities. An equivalence relation on a set S determines a partition of S, and conversely.

EXERCISES 4.1

★ 1. For each of the following binary relations ρ on $\mathbb{N}$, decide which of the given ordered pairs belong to ρ.

a. $x \rho y \leftrightarrow x + y < 7$; (1, 3), (2, 5), (3, 3), (4, 4)

b. $x \rho y \leftrightarrow x = y + 2$; (0, 2), (4, 2), (6, 3), (5, 3)

c. $x \rho y \leftrightarrow 2x + 3y = 10$; (5, 0), (2, 2), (3, 1), (1, 3)

d. $x \rho y \leftrightarrow y$ is a perfect square; (1, 1), (4, 2), (3, 9), (25, 5)

2. For each of the following binary relations ρ on $\mathbb{Z}$, decide which of the given ordered pairs belong to ρ.

a. $x \rho y \leftrightarrow x|y$; (2, 6), (3, 5), (8, 4), (4, 8)

b. $x \rho y \leftrightarrow x$ and y are relatively prime; (5, 8), (9, 16), (6, 8), (8, 21)

c. $x \rho y \leftrightarrow \gcd(x, y) = 7$; (28, 14), (7, 7), (10, 5), (21, 14)

d. $x \rho y \leftrightarrow x^2 + y^2 = z^2$ for some integer z; (1, 0), (3, 9), (2, 2), (3, 4)

e. $x \rho y \leftrightarrow x$ is a number from the Fibonacci sequence; (4, 3), (7, 6), (7, 12), (20, 20)

3. Decide which of the given items satisfy the relation.

a. ρ a binary relation on $\mathbb{Z}$, $x \rho y \leftrightarrow x = -y$; (1, −1), (2, 2), (−3, 3), (−4, −4)

b. ρ a binary relation on $\mathbb{N}$, $x \rho y \leftrightarrow x$ is prime; (19, 7), (21, 4), (33, 13), (41, 16)

c. ρ a binary relation on $\mathbb{Q}$, $x \rho y \leftrightarrow x \leq 1/y$; (1, 2), (−3, −5), (−4, 1/2), (1/2, 1/3)

d. ρ a binary relation on $\mathbb{N} \times \mathbb{N}$, $(x, y) \rho (u, v) \leftrightarrow x + u = y + v$; ((1, 2), (3, 2)), ((4, 5), (0, 1))

4. For each of the following binary relations on $\mathbb{R}$, draw a figure to show the region of the plane it describes.

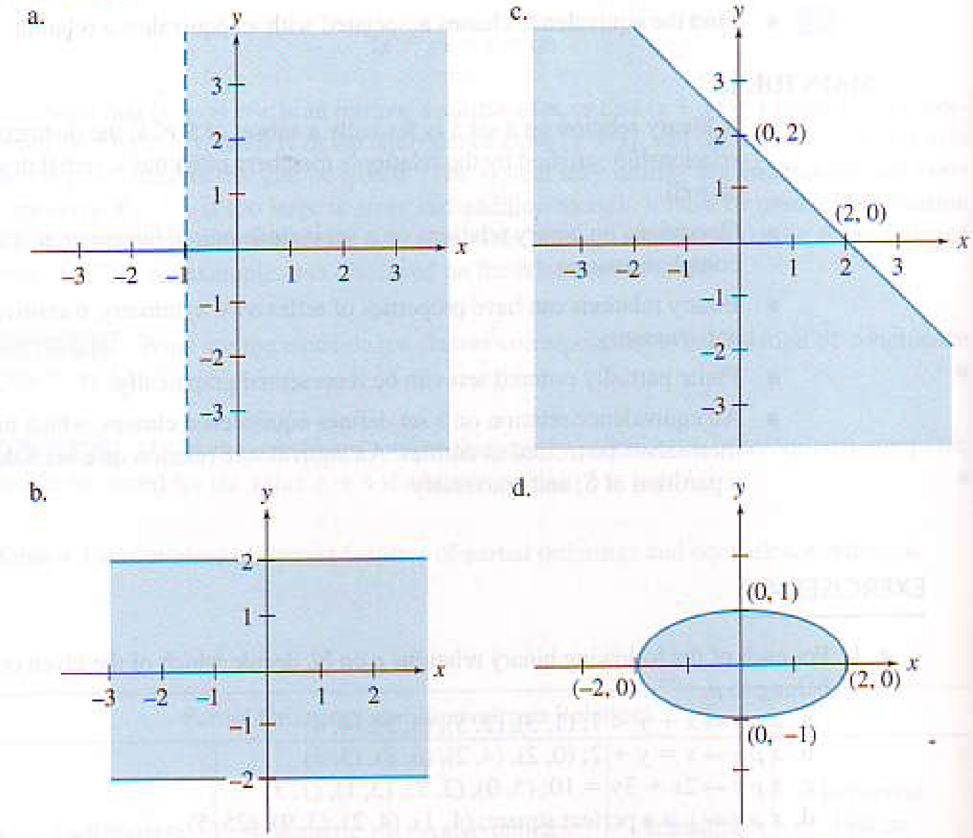
★ a. $x \rho y \leftrightarrow y \leq 2$

b. $x \rho y \leftrightarrow x = y - 1$

c. $x \rho y \leftrightarrow x^2 + y^2 \leq 25$

d. $x \rho y \leftrightarrow x \geq y$

5. For each of the accompanying figures, give the binary relation on $\mathbb{R}$ that describes the shaded area.



★ 6. Identify each relation on $\mathbb{N}$ as one-to-one, one-to-many, many-to-one, or many-to-many.

a. $\rho = \{(1, 2), (1, 4), (1, 6), (2, 3), (4, 3)\}$

b. $\rho = \{(9, 7), (6, 5), (3, 6), (8, 5)\}$

c. $\rho = \{(12, 5), (8, 4), (6, 3), (7, 12)\}$

d. $\rho = \{(2, 7), (8, 4), (2, 5), (7, 6), (10, 1)\}$

7. Identify each of the following relations on S as one-to-one, one-to-many, many-to-one, or many-to-many.

a. $S = \mathbb{N}$
$x \,\rho\, y \leftrightarrow x = y + 1$

b. S = set of all women in Vicksburg
$x \,\rho\, y \leftrightarrow x$ is the daughter of y

c. $S = \wp(\{1, 2, 3\})$
$A \,\rho\, B \leftrightarrow |A| = |B|$

d. $S = \mathbb{R}$
$x \,\rho\, y \leftrightarrow x = 5$

★ 8. Let ρ and σ be binary relations on $\mathbb{N}$ defined by $x \,\rho\, y \leftrightarrow$ "x divides y," $x \,\sigma\, y \leftrightarrow 5x \le y$. Decide which of the given ordered pairs satisfy the following relations:

a. $\rho \cup \sigma$; (2, 6), (3, 17), (2, 1), (0, 0)

b. $\rho \cap \sigma$; (3, 6), (1, 2), (2, 12)

c. ρ'; (1, 5), (2, 8), (3, 15)

d. σ'; (1, 1), (2, 10), (4, 8)

9. Let $S = \{1, 2, 3\}$. Test the following binary relations on S for reflexivity, symmetry, antisymmetry, and transitivity.

a. $\rho = \{(1, 3), (3, 3), (3, 1), (2, 2), (2, 3), (1, 1), (1, 2)\}$

b. $\rho = \{(1, 1), (3, 3), (2, 2)\}$

c. $\rho = \{(1, 1), (1, 2), (2, 3), (3, 1), (1, 3)\}$

d. $\rho = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$

MAIN IDEAS

- PERT charts are diagrams of partially ordered sets representing tasks and prerequisites among tasks.
- A topological sort extends a partial ordering on a finite set to a total ordering.

EXERCISES 4.2

1. The following tasks are required in order to assemble a bicycle. As the manufacturer, you must write a list of sequential instructions for the buyer to follow. Will the sequential order given below work? Give another sequence that could be used.

Task	Prerequisite Tasks
1. Tightening frame fittings	None
2. Attaching handle bars to frame	1
3. Attaching gear mechanism	1
4. Mounting tire on wheel assembly	None
5. Attaching wheel assembly to frame	1, 4
6. Installing brake mechanism	2, 3, 5
7. Adding pedals	6
8. Attaching seat	1
9. Adjusting seat height	7, 8

★ 2. Construct a PERT chart from the following task table.

Task	Prerequisite Tasks	Time to Perform
A	E	3
B	C, D	5
C	A	2
D	A	6
E	None	2
F	A, G	4
G	E	4
H	B, F	1

3. Construct a PERT chart from the following task table.

Task	Prerequisite Tasks	Time to Perform
1	2	4
2	3	2
3	8	5
4	3	2
5	4, 7	2
6	5	1
7	3	3
8	None	5

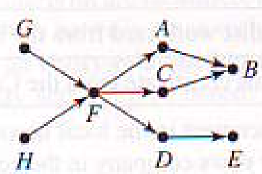
★ 4. Compute the minimum time to completion and the nodes on the critical path for the problem in Exercise 2.

5. Compute the minimum time to completion and the nodes on the critical path for the problem in Exercise 3.

6. For the problem in Exercise 2, great improvements in productivity have knocked down the time to perform task D from 6 units to 1 unit. Recompute the minimum time to completion and the nodes on the critical path.

7. For the problem in Exercise 3, an extra quality-control step has been added to task 4, which now requires 4 units of time to perform. Recompute the minimum time to completion and the nodes on the critical path.

8. Do a topological sort on the partially ordered set shown in the accompanying figure.



★ 9. Find a topological sort for the problem in Exercise 2.

10. Find a topological sort for the problem in Exercise 3.

SECTION 4.3 Review

TECHNIQUES

- W Carry out restrict, project, and join operations in a relational database.
- W Formulate relational database queries using relational algebra, SQL, and relational calculus.

MAIN IDEAS

- A relational database uses mathematical relations, described by tables, to model objects and relationships in an enterprise.
- The database operations of restrict, project, and join are operations on relations (sets of tuples).
- Queries on relational databases can be formulated using the restrict, project, and join operations, SQL statements, or notations borrowed from set theory and predicate logic.

EXERCISES 4.3

Exercises 1 and 2 refer to the Person and Pet relations of Example 20.

1. Consider the following operation:

   Restrict Pet **where** *Pet-type* = "Cat" **giving** Kitties.

   a. Write a query in English that would result in the information contained in Kitties.
   b. What is the cardinality of the relation obtained by performing this operation?
   c. Write an SQL query to obtain this information.

2. Consider the following operation:

   Project Person over (*Name, City, State*) **giving** Census.

   a. Write a query in English that would result in the information contained in Census.
   b. What is the degree of the relation obtained by performing this operation?
   c. Write an SQL query to obtain this information.

Exercises 3–21 are all related to the same enterprise.

★ 3. A library maintains a database about its books. Information kept on each author includes the author's name and country of origin and the titles of the author's books. Information kept on each book includes the title, ISBN, publisher, and subject. Authors and books are the entities in this enterprise, and "writes" is a relationship between these entities. Sketch an E-R diagram for the enterprise. In the absence of any business rules, what must be assumed about the binary relation "writes" regarding whether it is one-to-one, one-to-many, and so on?

4. Assume the business rule that authors are uniquely identified by their names. Does this change your answer to the last question of Exercise 3?

5. In a relational model of the library database, there is an author relation, a book relation, and a writes relation. Give the table heading for each of the relation tables, underlining the primary key. Explain your choice of primary keys. What implicit business rule (in addition to that of Exercise 4) underlies the choice of author attributes?

For Exercises 6–16, use the following relation tables and write the results of the operations.

Author

Name	*Country*	*Title*
Dorothy King	England	Springtime Gardening
Jon Nkoma	Kenya	Birds of Africa
Won Lau	China	Early Tang Paintings
Bert Kovalsco	U.S.	Baskets for Today
Tom Quercos	Mexico	Mayan Art
Jimmy Chan	China	Early Tang Paintings
Dorothy King	England	Autumn Annuals
Jane East	U.S.	Springtime Gardening

Book

Title	*ISBN*	*Publisher*	*Subject*
Springtime Gardening	0-816-35421-9	Harding	Nature
Early Tang Paintings	0-364-87547-X	Bellman	Art
Birds of Africa	0-115-01214-1	Loraine	Nature
Springtime Gardening	0-56-000142-8	Swift-Key	Nature
Baskets for Today	0-816-53705-4	Harding	Art
Autumn Annuals	0-816-88506-0	Harding	Nature

Writes

Name	*Title*	*ISBN*
Jimmy Chan	Early Tang Paintings	0-364-87547-X
Dorothy King	Autumn Annuals	0-816-88506-0
Jane East	Springtime Gardening	0-56-000142-8
Bert Kovalsco	Baskets for Today	0-816-53705-4
Won Lau	Early Tang Paintings	0-364-87547-X
Jon Nkoma	Birds of Africa	0-115-01214-1
Dorothy King	Springtime Gardening	0-816-35421-9

★ 6. **Restrict** Author **where** *Country* = "U.S." **giving** Results1.
7. **Restrict** Writes **where** *Name* = "Dorothy King" **giving** Results2.
8. **Restrict** Book **where** *Publisher* = "Bellman" or Publisher = "Swift-Key" **giving** Results3.
9. **Restrict** Book **where** *Publisher* = "Harding" and Subject = "Art" **giving** Results4.
★ 10. **Project** Author **over** (*Name, Title*) **giving** Results5.
11. **Project** Author **over** (*Name, Country*) **giving** Results6.
12. **Project** Book **over** (*Publisher, Subject*) **giving** Results7.
13. **Project** Book **over** (*Title, ISBN, Subject*) **giving** Results8.
★ 14. **Join** Book and Writes over *Title* and *ISBN* **giving** Results9.
15. **Join** Author and Writes over *Name* and *Title* **giving** Results10.
16. What would be wrong with doing a join of Author and Book over *Title*?

For Exercises 17–20, using the relation tables given, express each query in relational algebra, SQL, and relational calculus forms. Also give the result of each query.

★ 17. Give the titles of all books written by U.S. authors.
18. Give the names of all authors who publish with Harding.
19. Give the names of all authors who have written nature books.
20. Give the publishers of all art books whose authors live in the United States.
21. If the tuple

Suzanne Fleur, NULL, Perennials and You

is added to the Author table, write the results of the SQL query

SELECT Name
FROM Author
WHERE Author.Country = "U.S."
OR (Author.Country = NULL)

★ 22. Suppose a join operation over some attribute is to be done on two tables of cardinality p and q, respectively.
   a. The first step is usually to form the Cartesian product of the two relations and then examine the resulting tuples to find those with a common attribute value. How many tuples result from the Cartesian product that then have to be examined to complete the join operation?
   b. Now suppose that the two tables have each been sorted on the common attribute. Explain how the join operation can be done more cleverly, avoid the Cartesian product, and only examine at most $(p + q)$ rows.

SECTION 4.4 Functions

In this section we discuss functions, which are really special cases of binary relations from a set S to a set T. This view of a function is a rather sophisticated one, however, and we will work up to it gradually.

The big oh notation $f = O(g)$ says that f grows at the same rate or at a slower rate than g. But if we know that f definitely grows at a slower rate than g, then we can say something stronger, namely that f is **little oh** of g, written $f = o(g)$. The relationship between big oh and little oh is this: If $f = O(g)$, then either $f = \Theta(g)$ or $f = o(g)$.

SECTION 4.4 Review

TECHNIQUES

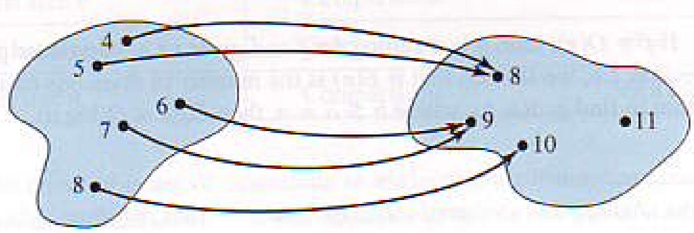
- Test whether a given relation is a function.
- W Test a function for being one-to-one or onto.
- Find the image of an element under function composition.
- W Write permutations of a set in array or cycle form.
- Count the number of functions, one-to-one functions, and onto functions from one finite set to another.
- Determine whether two functions are the same order of magnitude.

MAIN IDEAS

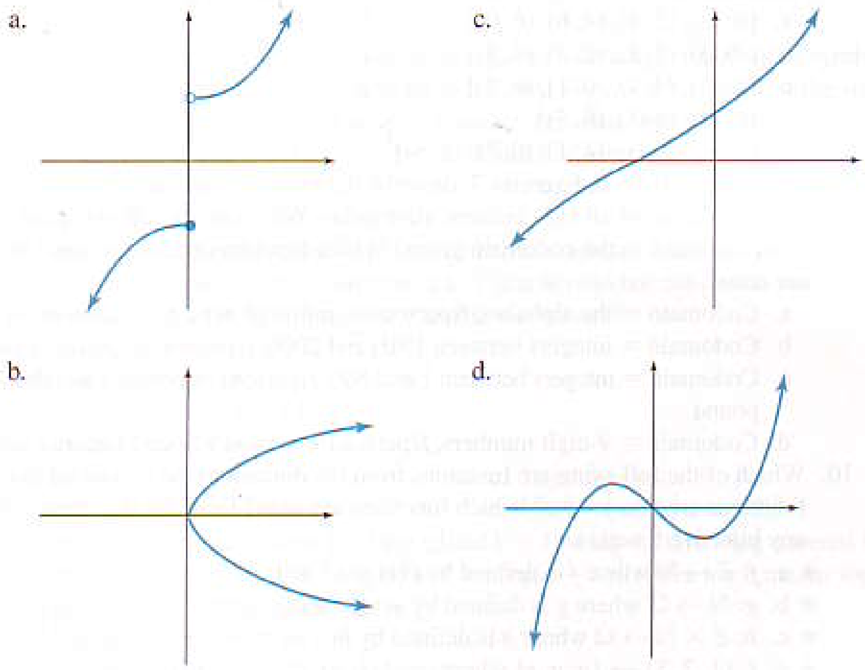
- The concept of function, especially bijective function, is extremely important.
- Composition of functions preserves bijectiveness.
- The inverse function of a bijection is itself a bijection.
- Permutations are bijections on a set.
- Functions can be grouped into equivalence classes according to their order of magnitude, which is a measure of their growth rate.

EXERCISES 4.4

★ 1. The accompanying figure represents a function.
   a. What is the domain? What is the codomain? What is the range?
   b. What is the image of 5? of 8?
   c. What are the preimages of 9?
   d. Is this an onto function? Is it one-to-one?



2. The accompanying figure illustrates various binary relations from $\mathbb{R}$ to $\mathbb{R}$. Which are functions? For those that are functions, which are onto? Which are one-to-one?



3. Using the equation $f(x) = 2x - 1$ to describe the functional association, write the function as a set of ordered pairs if the codomain is R and
   a. domain is $S = \{0, 1, 2\}$
   b. domain is $S = \{1, 2, 4, 5\}$
   c. domain is $S = \{\sqrt{7}, 1.5\}$

★ 4. If $f: \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = 3x$, find $f(A)$ for
   a. $A = \{1, 3, 5\}$
   b. $A = \{x \mid x \in \mathbb{Z} \text{ and } (\exists y)(y \in \mathbb{Z} \text{ and } x = 2y)\}$

5. If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, describe
   a. $f(\mathbb{N})$ b. $f(\mathbb{Z})$ c. $f(\mathbb{R})$

6. True or false:
   a. An onto function means that every element in the domain must have an image.
   b. An onto function means that every element in the codomain must have an image.
   c. An onto function means that every element in the codomain must have a preimage.
   d. An onto function means that every element in the codomain must have a unique preimage.
   e. A one-to-one function means that every element in the codomain must have a unique preimage.
   f. A one-to-one function means that no two elements in the domain map to the same element in the codomain.
   g. An onto function means that (the range) $\cap$ (the codomain) $= \varnothing$