15. The sum of an even integer and an odd integer is odd.
16. An odd integer minus an even integer is odd.
17. The product of any two consecutive integers is even.
18. The sum of an integer and its square is even.
★ 19. The square of an even number is divisible by 4.
20. For every integer $n$, the number

$$3(n^2 + 2n + 3) - 2n^2$$

is a perfect square.
21. If a number $x$ is positive, so is $x + 1$ (do a proof by contraposition).
22. The number $n$ is an odd integer if and only if $3n + 5 = 6k + 8$ for some integer $k$.
23. The number $n$ is an even integer if and only if $3n + 2 = 6k + 2$ for some integer $k$.
★ 24. For $x$ and $y$ positive numbers, $x < y$ if and only if $x^2 < y^2$.
25. If $x^2 + 2x - 3 = 0$, then $x \neq 2$.
26. If $n$ is an even prime number, then $n = 2$.
★ 27. If two integers are each divisible by some integer $n$, then their sum is divisible by $n$.
28. If the product of two integers is not divisible by an integer $n$, then neither integer is divisible by $n$.
29. If $n$, $m$, and $p$ are integers and $n|m$ and $m|p$, then $n|p$.
30. If $n$, $m$, $p$, and $q$ are integers and $n|p$ and $m|q$, then $nm|pq$.
31. The sum of three consecutive integers is divisible by 3.
★ 32. The square of an odd integer equals $8k + 1$ for some integer $k$.
33. The difference of two consecutive cubes is odd.
34. The sum of the squares of two odd integers cannot be a perfect square. (*Hint:* Use Exercise 32.)
★ 35. The product of the squares of two integers is a perfect square.
36. For any two numbers $x$ and $y$, $|xy| = |x||y|$.
★ 37. For any two numbers $x$ and $y$, $|x + y| \leq |x| + |y|$.
38. The value $A$ is the average of the $n$ numbers $x_1, x_2, \ldots, x_n$. Prove that at least one of $x_1, x_2, \ldots, x_n$ is greater than or equal to $A$.
39. Suppose you were to use the steps of Example 11 to attempt to prove that $\sqrt{4}$ is not a rational number. At what point would the proof not be valid?
40. Prove that $\sqrt{3}$ is not a rational number.
41. Prove that $\sqrt{5}$ is not a rational number.
42. Prove that $\sqrt[3]{2}$ is not a rational number.
43. Prove that $\log_2 5$ is not a rational number ($\log_2 5 = x$ means $2^x = 5$).

For Exercises 44–65, prove or disprove the given statement.

44. 91 is a composite number.
★ 45. 297 is a composite number.
46. 83 is a composite number.
47. The difference between two odd integers is odd.
48. The difference between two even integers is even.
★ 49. The product of any three consecutive integers is even.
50. The sum of any three consecutive integers is even.

10. $T(1) = 1$
$T(2) = 2$
$T(3) = 3$
$T(n) = T(n-1) + 2T(n-2) + 3T(n-3)$ for $n > 3$

In Exercises 11–15, prove the given property of the Fibonacci numbers directly from the definition.

★ 11. $F(n+1) + F(n-2) = 2F(n)$ for $n \geq 3$
12. $F(n) = 5F(n-4) + 3F(n-5)$ for $n \geq 6$
13. $[F(n+1)]^2 = [F(n)]^2 + F(n-1)F(n+2)$ for $n \geq 2$
14. $F(n+3) = 2F(n+1) + F(n)$ for $n \geq 1$
15. $F(n+6) = 4F(n+3) + F(n)$ for $n \geq 1$

In Exercises 16–19, prove the given property of the Fibonacci numbers for all $n \geq 1$. (*Hint:* The first principle of induction will work.)

★ 16. $F(1) + F(2) + \cdots + F(n) = F(n+2) - 1$
17. $F(2) + F(4) + \cdots + F(2n) = F(2n+1) - 1$
18. $F(1) + F(3) + \cdots + F(2n-1) = F(2n)$
19. $[F(1)]^2 + [F(2)]^2 + \cdots + [F(n)]^2 = F(n)F(n+1)$

In Exercises 20–23, prove the given property of the Fibonacci numbers using the second principle of induction.

★ 20. Exercise 14
21. Exercise 15
22. $F(n) < 2^n$ for $n \geq 1$
23. $F(n) > \left(\frac{3}{2}\right)^{n-1}$ for $n \geq 6$
24. The values $p$ and $q$ are defined as follows:

$$p = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad q = \frac{1-\sqrt{5}}{2}$$

a. Prove that $1 + p = p^2$ and $1 + q = q^2$.
b. Prove that

$$F(n) = \frac{p^n - q^n}{p - q}$$

c. Use part (b) to prove that

$$F(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is a closed-form solution for the Fibonacci sequence.

## EXERCISES 3.2

★1. A frozen yogurt shop allows you to choose one flavor (vanilla, strawberry, lemon, cherry, or peach), one topping (chocolate shavings, crushed toffee, or crushed peanut brittle), and one condiment (whipped cream or shredded coconut). How many different desserts are possible?

★2. In Exercise 1, how many dessert choices do you have if you are allergic to strawberries and chocolate?

3. A video game on a microcomputer is begun by making selections from each of three menus. The first menu (number of players) has four selections, the second menu (level of play) has eight, and the third menu (speed) has six. In how many configurations can the game be played?

4. A multiple-choice exam has 20 questions, each with four possible answers, and 10 additional questions, each with five possible answers. How many different answer sheets are possible?

5. A user's password to access a computer system consists of three letters followed by two digits. How many different passwords are possible?

6. On the computer system of Exercise 5, how many passwords are possible if uppercase and lowercase letters can be distinguished?

★7. A telephone conference call is being placed from Central City to Booneville by way of Cloverdale. There are 45 trunk lines from Central City to Cloverdale and 13 from Cloverdale to Booneville. How many different ways can the call be placed?

8. $A$, $B$, $C$, and $D$ are nodes on a computer network. There are two paths between $A$ and $C$, two between $B$ and $D$, three between $A$ and $B$, and four between $C$ and $D$. Along how many routes can a message from $A$ to $D$ be sent?

★9. How many Social Security numbers are possible?

10. An apartment building purchases a new lock system for its 175 units. A lock is opened by punching in a two-digit code. Has the apartment management made a wise purchase?

★11. A palindrome is a string of characters that reads the same forward and backward. How many five-letter English language palindromes are possible?

12. How many three-digit numbers less than 600 can be made using the digits 8, 6, 4, and 2?

13. A binary logical connective can be defined by giving its truth table. How many different binary logical connectives are there?

Exercises 14–17 are related to Example 29.

★14. Show that the 4 juggling patterns of length 2 using at most 2 balls—(3, 1), (1, 3), (2, 2), and (1, 1)—have stack numbers of 21, 12, 22, and 11, respectively.

15. a. Using 3 balls, find the juggling pattern of length 2 shown in the table.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| G | G | B | B | R | R | G | G | B | B | R | R | … |

b. Find the stack number for this juggling pattern.

c. Given a stack number of 221, draw a table for the corresponding juggling pattern of length 3 using 2 balls.

16. a. What is the number of juggling patterns of length 2 using at most 3 balls?

b. Write the stack numbers for the patterns of part (a).

c. Write the tables for these juggling patterns.

17. What is the number of juggling patterns of length 3 using at most 4 balls?

46. In Example 69:
    a. What is the probability that a patient responds positively to compound $A$ given that he or she responds positively to $B$?
    b. What is the probability that a patient responds positively to either compound $A$ or compound $B$?
    c. What is the probability that a patient does not respond positively to either compound?

For Exercises 47–52, a student takes a true/false quiz with four questions, each equally likely to be either T or F.

47. What is the probability of getting question 1 correct?
★ 48. What is the probability of getting exactly one question wrong?
49. What is the probability of getting three or more questions correct?
50. What is the probability of getting the first two questions correct?
★ 51. What is the probability of getting the first two questions correct given that the answer to question 1 is correct?
52. What is the probability of getting all four questions correct given that the answers to the first two questions are correct?

For Exercises 53–59, a family has 3 children; boys and girls are equally likely offspring.

53. What is the probability that the oldest child is a boy?
★ 54. What is the probability of 2 boys and 1 girl?
55. What is the probability of no girls?
56. What is the probability of at least one girl?
57. What is the probability of 3 girls?
★ 58. What is the probability of 3 girls given that the first two are girls?
59. What is the probability of at least one boy and at least one girl given that there is at least 1 boy?

Exercises 60–61 refer to Bayes' theorem, as stated in Exercise 60.

60. Let $E_1, \ldots, E_n$ be disjoint events from a sample space $S$ whose union equals $S$. If $F$ is another event from $S$, then Bayes' theorem says that the probability of event $E_i$, $1 \le i \le n$, given event $F$, is

$$P(E_i \mid F) = \frac{P(F \mid E_i)P(E_i)}{\sum_{k=1}^{n} P(F \mid E_k)P(E_k)}$$

a. Use the definition of $P(E_i \mid F)$ and $P(F \mid E_i)$ to prove that

$$P(E_i \mid F) = \frac{P(F \mid E_i)P(E_i)}{P(F)}$$