EXERCISES 3.1

★ 1. Let $S = \{2, 5, 17, 27\}$. Which of the following are true?
   a. $5 \in S$ b. $2 + 5 \in S$ c. $\varnothing \in S$ d. $S \in S$

2. Let $B = \{x \mid x \in \mathbb{Q}$ and $-1 < x < 2\}$. Which of the following are true?
   a. $0 \in B$ b. $-1 \in B$ c. $-0.84 \in B$ d. $\sqrt{2} \in B$

3. How many different sets are described in the following? What are they?

   $\{2, 3, 4\}$
   $\{x \mid x$ is the first letter of cat, bat, or apple$\}$
   $\{x \mid x \in \mathbb{N}$ and $2 \le x \le 4\}$
   $\{a, b, e\}$
   $\varnothing$
   $\{x \mid x$ is the first letter of cat, bat, and apple$\}$
   $\{2, a, 3, b, 4, c\}$
   $\{3, 4, 2\}$

★ 4. Describe each of the following sets by listing its elements:
   a. $\{x \mid x \in \mathbb{N}$ and $x^2 < 25\}$
   b. $\{x \mid x \in \mathbb{N}$ and x is even and $2 < x < 11\}$
   c. $\{x \mid x$ is one of the first three U.S. presidents$\}$
   d. $\{x \mid x \in \mathbb{R}$ and $x^2 = -1\}$
   e. $\{x \mid x$ is one of the New England states$\}$
   f. $\{x \mid x \in \mathbb{Z}$ and $|x| < 4\}$ ($|x|$ denotes the absolute value function)

5. Describe each of the following sets by listing its elements:
   a. $\{x \mid x \in \mathbb{N}$ and $x^2 - 5x + 6 = 0\}$
   b. $\{x \mid x \in \mathbb{R}$ and $x^2 = 7\}$
   c. $\{x \mid x \in \mathbb{N}$ and $x^2 - 2x - 8 = 0\}$

6. Describe each of the following sets by giving a characterizing property:
   a. $\{1, 2, 3, 4, 5\}$
   b. $\{1, 3, 5, 7, 9, 11, \ldots\}$
   c. {Melchior, Gaspar, Balthazar}
   d. $\{0, 1, 10, 11, 100, 101, 110, 111, 1000, \ldots\}$

7. Describe each of the following sets:
   a. $\{x \mid x \in \mathbb{N}$ and $(\exists q)(q \in \{2, 3\}$ and $x = 2q)\}$
   b. $\{x \mid x \in \mathbb{N}$ and $(\exists y)(\exists z)(y \in \{0, 1\}$ and $z \in \{3, 4\}$ and $y < x < z)\}$
   c. $\{x \mid x \in \mathbb{N}$ and $(\forall y)(y$ even $\rightarrow x \neq y)\}$

★ 8. Given the description of a set A as $A = \{2, 4, 8, \ldots\}$, do you think $16 \in A$?

9. What is the cardinality of each of the following sets?
   a. $S = \{a, \{a, \{a\}\}\}$
   b. $S = \{\{a\}, \{\{a\}\}\}$
   c. $S = \{\varnothing\}$
   d. $S = \{a, \{\varnothing\}, \varnothing\}$
   e. $S = \{\varnothing, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing, \{\varnothing\}\}\}\}$

★ 10. Let

$$A = \{2, 5, 7\}$$
$$B = \{1, 2, 4, 7, 8\}$$
$$C = \{7, 8\}$$

Which of the following statements are true?
   a. $5 \subseteq A$ b. $C \subseteq B$ c. $\varnothing \in A$ d. $7 \in B$ e. $\{2, 5\} \subseteq A$ f. $\varnothing \subseteq C$

EXERCISES 3.2

★1. A frozen yogurt shop allows you to choose one flavor (vanilla, strawberry, lemon, cherry, or peach), one topping (chocolate shavings, crushed toffee, or crushed peanut brittle), and one condiment (whipped cream or shredded coconut). How many different desserts are possible?

★2. In Exercise 1, how many dessert choices do you have if you are allergic to strawberries and chocolate?

3. A video game on a microcomputer is begun by making selections from each of three menus. The first menu (number of players) has four selections, the second menu (level of play) has eight, and the third menu (speed) has six. In how many configurations can the game be played?

4. A multiple-choice exam has 20 questions, each with four possible answers, and 10 additional questions, each with five possible answers. How many different answer sheets are possible?

5. A user's password to access a computer system consists of three letters followed by two digits. How many different passwords are possible?

6. On the computer system of Exercise 5, how many passwords are possible if uppercase and lowercase letters can be distinguished?

★7. A telephone conference call is being placed from Central City to Booneville by way of Cloverdale. There are 45 trunk lines from Central City to Cloverdale and 13 from Cloverdale to Booneville. How many different ways can the call be placed?

8. A, B, C, and D are nodes on a computer network. There are two paths between A and C, two between B and D, three between A and B, and four between C and D. Along how many routes can a message from A to D be sent?

★9. How many Social Security numbers are possible?

10. An apartment building purchases a new lock system for its 175 units. A lock is opened by punching in a two-digit code. Has the apartment management made a wise purchase?

★11. A palindrome is a string of characters that reads the same forward and backward. How many five-letter English language palindromes are possible?

12. How many three-digit numbers less than 600 can be made using the digits 8, 6, 4, and 2?

13. A binary logical connective can be defined by giving its truth table. How many different binary logical connectives are there?

Exercises 14–17 are related to Example 29.

★14. Show that the 4 juggling patterns of length 2 using at most 2 balls—(3, 1), (1, 3), (2, 2), and (1, 1)—have stack numbers of 21, 12, 22, and 11, respectively.

15. a. Using 3 balls, find the juggling pattern of length 2 shown in the table.

G	G	B	B	R	R	G	G	B	B	R	R	...

b. Find the stack number for this juggling pattern.

c. Given a stack number of 221, draw a table for the corresponding juggling pattern of length 3 using 2 balls.

16. a. What is the number of juggling patterns of length 2 using at most 3 balls?

b. Write the stack numbers for the patterns of part (a).

c. Write the tables for these juggling patterns.

17. What is the number of juggling patterns of length 3 using at most 4 balls?

- Use the pigeonhole principle to find the minimum number of elements to guarantee two with a duplicate property.

MAIN IDEA

- The principle of inclusion and exclusion and the pigeonhole principle are additional counting mechanisms for sets.

EXERCISES 3.3

1. In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 18 French speakers. How many speak both English and French?

★ 2. All the guests at a dinner party drink coffee or tea; 13 guests drink coffee, 10 drink tea, and 4 drink both coffee and tea. How many people are in this group?

3. After serving 137 customers, a cafeteria notes at the end of the day that 56 orders of green beans were sold, 38 orders of beets were sold, and 17 customers purchased both green beans and beets. How many customers bought neither beans nor beets?

4. Quality control in a factory pulls 40 parts with paint, packaging, or electronics defects from an assembly line. Of these, 28 had a paint defect, 17 had a packaging defect, 13 had an electronics defect, 6 had both paint and packaging defects, 7 had both packaging and electronics defects, and 10 had both paint and electronics defects. Did any part have all three types of defect?

5. In a group of 24 people who like rock, country, and classical music, 14 like rock, 17 like classical, 11 like both rock and country, 9 like rock and classical, 13 like country and classical, and 8 like rock, country, and classical. How many like country?

★ 6. Nineteen different mouthwash products make the following claims: 12 claim to freshen breath, 10 claim to prevent gingivitis, 11 claim to reduce plaque, 6 claim to both freshen breath and reduce plaque, 5 claim to both prevent gingivitis and freshen breath, and 5 claim to both prevent gingivitis and reduce plaque.
   a. How many products make all three claims?
   b. How many products claim to freshen breath but do not claim to reduce plaque?

7. From the 83 students who want to enroll in CS 320, 32 have completed CS 120, 27 have completed CS 180, and 35 have completed CS 215. Of these, 7 have completed both CS 120 and CS 180, 16 have completed CS 180 and CS 215, and 3 have completed CS 120 and CS 215. Two students have completed all three courses. The prerequisite for CS 320 is completion of one of CS 120, CS 180, or CS 215. How many students are not eligible to enroll?

8. Among a bank's 214 customers with checking or savings accounts, 189 have checking accounts, 73 have regular savings accounts, 114 have money market savings accounts, and 69 have both checking and regular savings accounts. No customer is allowed to have both regular savings and money market savings accounts.
   a. How many customers have both checking and money market savings accounts?
   b. How many customers have a checking account but no savings account?

★ 9. A survey of 150 college students reveals that 83 own automobiles, 97 own bikes, 28 own motorcycles, 53 own a car and a bike, 14 own a car and a motorcycle, 7 own a bike and a motorcycle, and 2 own all three.
   a. How many students own a bike and nothing else?
   b. How many students do not own any of the three?

SECTION 3.4 Review

TECHNIQUES

- Find the number of permutations of r distinct objects chosen from n distinct objects.
- Find the number of combinations of r distinct objects chosen from n distinct objects.
- W Use permutations and combinations in conjunction with the multiplication principle and the addition principle.
- Find the number of distinct permutations of n objects that are not all distinct.
- Find the number of permutations of r objects out of n distinct objects when objects may be repeated.
- Find the number of combinations of r objects out of n distinct objects when objects may be repeated.
- Generate all permutations of the integers $\{1, \ldots, n\}$ in lexicographical order.
- Generate all combinations of r integers from the set $\{1, \ldots, n\}$.

MAIN IDEAS

- There are formulas for counting various permutations and combinations of objects.
- Care must be taken when analyzing a counting problem to avoid counting the same thing more than once or not counting some things at all.
- Algorithms exist to generate all permutations of n objects and all combinations of r out of n objects.

EXERCISES 3.4

1. Compute the value of the following expressions.
   ★ a. $P(7, 2)$ b. $P(8, 5)$ c. $P(6, 4)$ d. $P(n, n - 1)$
2. How many batting orders are possible for a nine-man baseball team?
3. The 14 teams in the local Little League are listed in the newspaper. How many listings are possible?
4. How many permutations of the characters in COMPUTER are there? How many of these end in a vowel?
5. ★ How many distinct permutations of the characters in ERROR are there? (Remember that the various R's cannot be distinguished from one another.)
6. In how many ways can six people be seated in a circle of six chairs? Only relative positions in the circle can be distinguished.
7. ★ In how many ways can first, second, and third prize in a pie-baking contest be given to 15 contestants?
8. a. Stock designations on an exchange are limited to three letters. How many different designations are there?
   b. How many different designations are there if letters cannot be repeated?
9. In how many different ways can you seat 11 men and 8 women in a row?
10. ★ In how many different ways can you seat 11 men and 8 women in a row if the men all sit together and the women all sit together?

MAIN IDEAS

- An event is a subset of the set of all possible outcomes of some action.
- For equally likely outcomes, the probability of an event is the ratio of the number of outcomes in the event to the number of all possible outcomes.
- In the conditional probability of event E_2 given that event E_1 has already taken place, the sample space is reduced to E_1.
- Events E_1 and E_2 are independent if and only if the conditional probability of E_2 given E_1 is the same as the probability of E_2.
- Given a sample space to which a random variable and a probability distribution have been assigned, the expected value of the random variable is a predictor of its future value.
- An average case analysis of an algorithm is the expected value of work units over the sample space of all inputs; the probability distribution reflects the assumptions being made about "average" input.

EXERCISES 3.5

Exercises 1–5 concern three coins tossed at the same time, each equally likely to come up heads or tails.

★ 1. What is the size of the sample space?
2. What is the probability of getting one heads and two tails?
3. What is the probability of getting all tails?
★ 4. What is the probability of getting all tails or all heads?
5. What is the probability of getting all tails and all heads?

In Exercises 6–12, a pair of fair dice is rolled.

★ 6. What is the size of the sample space?
7. What is the probability of getting "snake eyes" (two 1s)?
8. What is the probability of getting doubles (the same number on each die?
★ 9. What is the probability of getting a total of 7 on the two dice?
10. What is the probability of getting a 1 on at least one die?
11. What is the probability of getting a total on the two dice greater than 10?
12. What is the probability of getting a total on the two dice that is an odd number?

Exercises 13–16 concern three people participating in a race, with each participant equally likely to either finish the race or drop out of the race.

13. What is the size of the sample space?
14. What is the probability of exactly one participant finishing the race?
15. What is the probability of no one finishing the race?
16. What is the probability of at least two of the participants finishing the race?

Exercises 17–24 concern 2-card hands from a standard 52-card deck. A standard deck has 13 cards from each of 4 suits (clubs, diamonds, hearts, spades). The 13 cards have face value 2 through 10, jack, queen, king, or ace. Each face value is a "kind" of card. The jack, queen, and king are "face cards."

★ 17. What is the size of the sample space?