MAIN IDEAS

- Inductive reasoning is used to formulate a conjecture based on experience.
- Deductive reasoning is used either to refute a conjecture by finding a counterexample or to prove a conjecture.
- In proving a conjecture about some subject, facts about that subject can be used.
- Under the right circumstances, proof by contraposition or contradiction may work better than a direct proof.

EXERCISES 2.1

★ 1. Write the converse and the contrapositive of each statement in Exercise 4 of Section 1.1.

2. Given the implication $P \to Q$, then $Q' \to P'$ is the contrapositive of the implication and $Q \to P$ is the converse of the implication. The only remaining variation is the *inverse* of the implication, defined as $P' \to Q'$. To which of the other three (implication, contrapositive, converse) is the inverse equivalent?

3. Provide counterexamples to the following statements.
   a. Every geometric figure with four right angles is a square.
   b. If a real number is not positive, then it must be negative.
   c. All people with red hair have green eyes or are tall.
   d. All people with red hair have green eyes and are tall.

4. Provide counterexamples to the following statements.
   a. The number n is an odd integer if and only if $3n + 5$ is an even integer.
   b. The number n is an even integer if and only if $3n + 2$ is an even integer.

5. a. Find two even integers whose sum is not a multiple of 4.
   b. What is wrong with the following "proof" that the sum of two even numbers is a multiple of 4?

   Let x and y be even numbers. Then $x = 2m$ and $y = 2m$, where m is an integer, so $x + y = 2m + 2m = 4m$, which is an integral multiple of 4.

6. a. Find an example of an odd number x and an even number y such that $x - y = 7$.
   b. What is wrong with the following "proof" that an odd number minus an even number is always 1?

   Let x be odd and y be even. Then $x = 2m + 1$, $y = 2m$, where m is an integer, and $x - y = 2m + 1 - 2m = 1$.

For Exercises 7–43, prove the given statement.

★ 7. If $n = 25$, 100, or 169, then n is a perfect square and is a sum of two perfect squares.

8. If n is an even integer, $4 \le n \le 12$, then n is a sum of two prime numbers.

9. For any positive integer n less than or equal to 3, $n! < 2^n$.

10. For $2 \le n \le 4$, $n^2 \ge 2^n$.

11. 0 is an even number.

12. The sum of even integers is even (do a direct proof).

13. The sum of even integers is even (do a proof by contradiction).

★ 14. The sum of two odd integers is even.

15. The sum of an even integer and an odd integer is odd.
16. An odd integer minus an even integer is odd.
17. The product of any two consecutive integers is even.
18. The sum of an integer and its square is even.
★ 19. The square of an even number is divisible by 4.
20. For every integer n, the number

$$3(n^2 + 2n + 3) - 2n^2$$

is a perfect square.
21. If a number x is positive, so is $x + 1$ (do a proof by contraposition).
22. The number n is an odd integer if and only if $3n + 5 = 6k + 8$ for some integer k.
23. The number n is an even integer if and only if $3n + 2 = 6k + 2$ for some integer k.
★ 24. For x and y positive numbers, $x < y$ if and only if $x^2 < y^2$.
25. If $x^2 + 2x - 3 = 0$, then $x \neq 2$.
26. If n is an even prime number, then $n = 2$.
★ 27. If two integers are each divisible by some integer n, then their sum is divisible by n.
28. If the product of two integers is not divisible by an integer n, then neither integer is divisible by n.
29. If n, m, and p are integers and $n|m$ and $m|p$, then $n|p$.
30. If n, m, p, and q are integers and $n|p$ and $m|q$, then $nm|pq$.
31. The sum of three consecutive integers is divisible by 3.
★ 32. The square of an odd integer equals $8k + 1$ for some integer k.
33. The difference of two consecutive cubes is odd.
34. The sum of the squares of two odd integers cannot be a perfect square. (*Hint:* Use Exercise 32.)
★ 35. The product of the squares of two integers is a perfect square.
36. For any two numbers x and y, $|xy| = |x||y|$.
★ 37. For any two numbers x and y, $|x + y| \leq |x| + |y|$.
38. The value A is the average of the n numbers $x_1, x_2, \ldots, x_n$. Prove that at least one of $x_1, x_2, \ldots, x_n$ is greater than or equal to A.
39. Suppose you were to use the steps of Example 11 to attempt to prove that $\sqrt{4}$ is not a rational number. At what point would the proof not be valid?
40. Prove that $\sqrt{3}$ is not a rational number.
41. Prove that $\sqrt{5}$ is not a rational number.
42. Prove that $\sqrt[3]{2}$ is not a rational number.
43. Prove that $\log_2 5$ is not a rational number ($\log_2 5 = x$ means $2^x = 5$).

For Exercises 44–65, prove or disprove the given statement.

44. 91 is a composite number.
★ 45. 297 is a composite number.
46. 83 is a composite number.
47. The difference between two odd integers is odd.
48. The difference between two even integers is even.
★ 49. The product of any three consecutive integers is even.
50. The sum of any three consecutive integers is even.