EXAMPLE 24 The wff

$$(\exists x)P(x) \rightarrow (\forall x)P(x)$$

is not valid. For example, in the interpretation where the domain consists of the integers and $P(x)$ means that x is even, it is true that there exists an integer that is even, but it is false that every integer is even. The antecedent of the implication is true and the consequent is false, so the value of the implication is false. ●

We do not necessarily have to go to a mathematical context to construct an interpretation in which a wff is false, but it is frequently easier to do so because the relationships among objects are relatively clear.

PRACTICE 21 Is the wff

$$(\forall x)[P(x) \vee Q(x)] \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$$

valid or invalid? Explain. ●

SECTION 1.3 Review

TECHNIQUES

- W ● Determine the truth value of a predicate wff in a given interpretation.
- W ● Translate English language statements into predicate wffs, and vice versa.
- ● Recognize a valid wff and explain why it is valid.
- W ● Recognize a nonvalid wff and construct an interpretation in which it is false or has no truth value.

MAIN IDEAS

- The truth value of predicate wffs depends on the interpretation considered.
- Valid predicate wffs are "intrinsically true"—true in all interpretations.

EXERCISES 1.3

1. What is the truth value of each of the following wffs in the interpretation where the domain consists of the integers, $O(x)$ is "x is odd," $L(x)$ is "$x < 10$," and $G(x)$ is "$x > 9$"?
   a. $(\exists x)O(x)$
   b. $(\forall x)[L(x) \rightarrow O(x)]$
   c. $(\exists x)[L(x) \wedge G(x)]$
   d. $(\forall x)[L(x) \vee G(x)]$
2. What is the truth value of each of the following wffs in the interpretation where the domain consists of the integers?
   ★ a. $(\forall x)(\exists y)(x + y = x)$
   ★ b. $(\exists y)(\forall x)(x + y = x)$
   ★ c. $(\forall x)(\exists y)(x + y = 0)$
   ★ d. $(\exists y)(\forall x)(x + y = 0)$
   e. $(\forall x)(\forall y)(x < y \vee y < x)$
   f. $(\forall x)[x < 0 \rightarrow (\exists y)(y > 0 \wedge x + y = 0)]$
   g. $(\exists x)(\exists y)(x^2 = y)$
   h. $(\forall x)(x^2 > 0)$

3. Give the truth value of each of the following wffs in the interpretation where the domain consists of the states of the United States, $Q(x, y)$ is "x is north of y," $P(x)$ is "x starts with the letter M," and a is "Massachusetts."
   a. $(\forall x)P(x)$
   b. $(\forall x)(\forall y)(\forall z)[Q(x, y) \wedge Q(y, z) \rightarrow Q(x, z)]$
   c. $(\exists y)(\exists x)Q(y, x)$
   d. $(\forall x)(\exists y)[P(y) \wedge Q(x, y)]$
   e. $(\exists y)Q(a, y)$
   f. $(\exists x)[P(x) \wedge Q(x, a)]$
4. Give the truth value of each of the following wffs in the interpretation where the domain consists of people: $M(x, y)$ is "x is the mother of y," $F(x)$ is "x is female," $M(x)$ is "x is male."
   a. $(\forall x)(\exists y)(M(y, x))$
   b. $(\exists x)(\forall y)(M(x, y))$
   c. $(\forall x)(\forall y)(M(x, y) \rightarrow M(y))$
   d. $(\exists x)(\exists y)(M(x, y) \wedge M(y))$
   e. $(\exists x)(\forall y)(M(x, y) \rightarrow F(y))$
5. For each wff, find an interpretation in which it is true and one in which it is false.
   ★ a. $(\forall x)([A(x) \vee B(x)] \wedge [A(x) \wedge B(x)]')$
   b. $(\forall x)(\forall y)[P(x, y) \rightarrow P(y, x)]$
   c. $(\forall x)[P(x) \rightarrow (\exists y)Q(x, y)]$
   d. $(\exists x)[A(x) \wedge (\forall y)B(x, y)]$
   e. $[(\forall x)A(x) \rightarrow (\forall x)B(x)] \rightarrow (\forall x)[A(x) \rightarrow B(x)]$
6. Identify the scope of each of the quantifiers in the following wffs and indicate any free variables.
   a. $(\forall x)[P(x) \rightarrow Q(y)]$
   b. $(\exists x)[A(x) \wedge (\forall y)B(y)]$
   c. $(\exists x)[(\forall y)P(x, y) \wedge Q(x, y)]$
   d. $(\exists x)(\exists y)[A(x, y) \wedge B(y, z) \rightarrow A(a, z)]$
7. Which of the following are equivalent to the statement

   All circles are round.

   a. If it's round, it's a circle.
   b. Roundness is a necessary property of circles.
   c. Something that isn't round can't be a circle.
   d. Some round things are circles.
8. Which of the following are equivalent to the statement

   Cats are smarter than dogs.

   a. Some cats are smarter than some dogs.
   b. There is a cat that is smarter than all dogs.
   c. All cats are smarter than all dogs.
   d. Only cats are smarter than dogs.
   e. All cats are smarter than any dog.
9. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

   $D(x)$ is "x is a day."
   $S(x)$ is "x is sunny."
   $R(x)$ is "x is rainy."
   M is "Monday."
   T is "Tuesday."

★ a. All days are sunny.
★ b. Some days are not rainy.
★ c. Every day that is sunny is not rainy.
d. Some days are sunny and rainy.
e. No day is both sunny and rainy.
f. It is always a sunny day only if it is a rainy day.
g. No day is sunny.
h. Monday was sunny; therefore every day will be sunny.
i. It rained both Monday and Tuesday.
j. If some day is rainy, then every day will be sunny.

10. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$B(x)$ is "x is a ball."
$R(x)$ is "x is round."
$S(x)$ is "x is a soccer ball."

a. All balls are round.
b. Not all balls are soccer balls.
c. All soccer balls are round.
d. Some balls are not round.
e. Some balls are round but soccer balls are not.
f. Every round ball is a soccer ball.
g. Only soccer balls are round balls.
h. If soccer balls are round, then all balls are round.

11. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$P(x)$ is "x is a person."
$T(x)$ is "x is a time."
$F(x, y)$ is "x is fooled at y."

a. You can fool some of the people all of the time.
b. You can fool all of the people some of the time.
c. You can't fool all of the people all of the time.

12. Using the predicate symbols shown and appropriate quantifiers, write each English language statement as a predicate wff. (The domain is the whole world.)

$L(x)$: x is a lion.
$R(x)$: x roars.
$P(x)$: x is a predator.
$Z(x)$: x is a zebra.
$E(x, y)$: x eats y.

a. All lions are predators.
b. Some lions roar.
c. Only lions roar.
d. Some lions eat all zebras.
e. All lions eat all zebras.

19. Three forms of negation are given for each statement. Which is correct?
    - ★ a. Some people like mathematics.
        1. Some people dislike mathematics.
        2. Everybody dislikes mathematics.
        3. Everybody likes mathematics.
    - b. Everyone loves ice cream.
        1. No one loves ice cream.
        2. Everyone dislikes ice cream.
        3. Someone doesn't love ice cream.
    - c. All people are tall and thin.
        1. Someone is short and fat.
        2. No one is tall and thin.
        3. Someone is short or fat.
    - d. Some pictures are old or faded.
        1. Every picture is neither old nor faded.
        2. Some pictures are not old or faded.
        3. All pictures are not old or not faded.
20. Three forms of negation are given for each statement. Which is correct?
    - a. Nobody is perfect.
        1. Everyone is imperfect.
        2. Everyone is perfect.
        3. Someone is perfect.
    - b. All swimmers are tall.
        1. Some swimmer is not tall.
        2. There are no tall swimmers.
        3. Every swimmer is short.
    - c. Every planet is cold and lifeless.
        1. No planet is cold and lifeless.
        2. Some planet is not cold and not lifeless.
        3. Some planet is not cold or not lifeless.
    - d. No bears are hungry.
        1. Only bears are hungy.
        2. All bears are hungry.
        3. There is a hungry bear.
21. Write the negation of each of the following.
    - a. Some Web sites feature audio.
    - b. Every Web site has both audio and video.
    - ★ c. Every Web site has either audio or video.
    - d. Some Web sites have neither audio nor video.
    - e. Every Web site either has text or else has both audio and video.
22. Write the negation of each of the following.
    - a. Only students eat pizza.
    - b. Every student eats pizza
    - c. Some students eat only pizza.
23. Write the negation of each of the following.
    - a. Some farmer grows only corn.
    - b. All farmers grow corn.
    - c. Corn is grown only by farmers.